

Finite Semihypergroups Built From Groups

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Abstract

Necessary and sufficient conditions for finite semihypergroups to be built from groups of the same order are established.

Introduction The theory of hypergroups and semihypergroups was introduced by C. Dunkl [1], R. Jewett [2], and R. Spector [3] and is well developed now.

Many examples of finite commutative semihypergroups and hypergroups can be found in [4]. In [4] there is a precise *physical* definition of a finite semihypergroup - commutative semihypergroup because this example describes a finite collection of particles which interact by colliding.

Here is a more suitable example. Let us say we are observing a finite collection of events $\{e_1, \dots, e_n\}$, which may be in a cause and effect relationship. To establish that we will build a cube with frequencies (probabilities) $a_{i,j}(k)$ in which each event e_k appears (be third) after any sequential pair of events e_i and e_j in our observed (infinitely long) sequence of events. So, we have a Markov chain of second order. Now assume that convolution - $e_i * e_j = \sum_k a_{i,j}(k)e_k$ - is associative and this semihypergroup (cube) can be developed from a group. In terms of this example the main result of present article shows that for such semihypergroups all observed events are not simple but rather are a combination of some simple events, evolution of which can be described as a group product for some appropriate group.

Let \mathbf{H} be a finite semihypergroup with n states e_1, e_2, \dots, e_n and associative convolution operation defined by

$$e_i * e_j = \sum_k a_{i,j}(k)e_k \quad (i, j, k = 1, 2, \dots, n) \quad (1)$$

where $a_{i,j}(k) \geq 0$ and $\sum_{k=1}^n a_{i,j}(k) = 1$ for each i, j .

Let us denote

columns $\{a_{i,j}(1), \dots, a_{i,j}(n)\}$ by $a_{i,j}$

matrix with columns $\{a_{i,1}, \dots, a_{i,n}\}$ by A_i (matrix of left regular representation)

matrix with columns $\{a_{1,i}, \dots, a_{n,i}\}$ by B_i (matrix of right regular representation)

and cube with matrices $\{A_1, \dots, A_n\}$ by C .

Definition A semihypergroup \mathbf{H} will be said to be *derived from a group* if any of the following two conditions is satisfied:

- (A) There are only n different rows in all A_i and only n different rows in all B_i and in any A_i rows are linearly independent and in any B_i rows are linearly independent.
- (B) There exists a group \mathbf{G} of order n with elements $\{g_1, \dots, g_n\}$ and measure m on \mathbf{G} that $e_i = m \cdot g_i$ where $g_i \cdot g_j$ is a group product and

$$e_i * e_j = (m \cdot g_i) \cdot (m \cdot g_j)$$

Now let us establish some properties which are shared among all semihypergroups with condition (A).

In matrix terms the convolution operation (1) transforms to $a_{i,j} = A_i e_j$ (where e_j is a column-vector with 1 on j -th position and 0 otherwise). The matrix $\sum c_i A_i$ represents a measure $\sum c_i e_i$ so $\sum_{k=1}^n a_{i,j}(k) A_k$ represents $e_i * e_j$. Now it is obvious that the convolution operation (1) is associative if and only if

$$A_i A_j = \sum_{k=1}^n a_{i,j}(k) A_k \quad (2)$$

Theorem 1. If a semihypergroup satisfies condition (A) then there exist n different non negative numbers among all $a_{i,j}(k)$ that $\sum a_k = 1$ and for each a_p from $\{a_1, \dots, a_n\}$ and each $i, j = 1, \dots, n$ there exist such k and l that

$$a_{i,k}(j) = a_p \text{ and } a_{l,i}(j) = a_p$$

Let us denote as $r_{i,j}$ j -th row in A_i . Then in $A_1 A_1$ the first row is $\sum r_{1,1}(k) r_{1,k}$ and because of (2) it is $\sum a_{1,1} r_{k,1}$. Now because $r_{1,k}(k = 1, \dots, n)$ are linearly independent we conclude that $\{r_{k,1}\}$ contains the same set of linearly independent rows as $\{r_{1,k}\}$ and the first column $a_{1,1}$ contains the same elements as the first row $r_{1,1}$.

The second row in $A_1 A_1$ is $\sum r_{1,2}(k) r_{1,k} = \sum a_{1,1}(k) r_{k,2}$. Again we conclude that $\{r_{k,2}\}$ contains the same set of linearly independent rows as $\{r_{1,k}\}$ and the row $r_{1,2}$ contains the same elements as the column $a_{1,1}$.

Continuing this consideration for all $A_i A_j$, we conclude that each row in each A_i and each column in C contains the same elements. Now, if we repeat this process with matrices B_i , we will see that each row in each matrix B_i contains the same elements as each column in C .

Now let us establish some properties which are shared among all semihypergroups derived from a group.

Corollary 1. Diagonal elements in each matrix A_i and B_i are equal.

The i -th row in $A_1 A_1$ is

$$\sum r_{1,i}(k) r_{1,k} = \sum a_{1,1} r_{k,i}$$

when here in the left part $k = i$ and in the right part $k = 1$ we have $r_{1,i}(i) = a_{1,1}(1) = r_{1,1}(1)$.

Corollary 2. Every matrix A_i is a linear combination of G_i , where $\{G_i\}$ - matrices of a left regular representation of some group \mathbf{G} of order n .

To prove this we will use the following alternative definition of a group:

- a) On non-empty set G of elements $\{g_i\}$ with a binary operation - $g_i \cdot g_j = g_k$;
- b) This operation is associative;
- c) For any elements g_i and g_j there exist at least one such g_k and at least one such g_l that

$$g_k \cdot g_i = g_i \cdot g_l = g_j.$$

By condition (B) there are no more than n different elements in C . Let us assume that all n elements are different and one of them is a .

Then for any i, k there exists such j that

$$a_{i,j}(k) = a \quad (3)$$

and for any j, k there exists such i that

$$a_{i,j}(k) = a \quad (4)$$

When $a = 1$ and all other elements in the cube C are equal to 0, conditions (3) and (4) describe binary operation defined as

$$g_i \cdot g_j = a_{i,j}(k) = g_k \quad (5)$$

Let us show that the operation (5) is associative.

$$(e_i * e_j) * e_m = \sum_{k=1}^n a_{i,j}(k)(e_k * e_m) = \sum_{k=1}^n a_{i,j}(k)a_{k,m} \quad (6)$$

On the other hand,

$$e_i * (e_j * e_m) = \sum_{p=1}^n a_{j,m}(p)(e_i * e_p) = \sum_{p=1}^n a_{j,m}(p)a_{i,p} \quad (7)$$

Now from (6) and (7)

$$\sum a_{i,j}(k)a_{k,m} = \sum a_{j,m}(p)a_{i,p}$$

and because $a_{i,j}(k) = a_{j,m}(p) = 1$ that means that

$$(g_i \cdot g_j) \cdot g_m = g_i \cdot (g_j \cdot g_m).$$

Therefore, a cube C with only one element a_p equal to 1 and all other elements equals to 0, describes a group.

Let us show that different a_i, a_j describe the same group. Assume that a_i and a_j correspondent to two different groups with its left representations G_1 and G_2 .

Consider the following two fusions of their matrices

$$G_{1,i_1}, \dots, G_{1,i_k}, \dots, G_{1,i_n} \quad (8)$$

$$G_{1,j_1}, \dots, G_{1,j_k}, \dots, G_{1,j_n} \quad (9)$$

$$G_{2,j_1}, \dots, G_{2,j_k}, \dots, G_{2,j_n} \quad (10)$$

Here in (8) and (9) we have matrices from representation of the same group in different order when in pair (9) and (10) there are matrices from representation of different groups and $G_{1,j_k} \neq G_{2,j_k}$. Then (let us assume there are only two numbers a_1 and a_2 not equal to zero) if matrices in (8) and (9) are in such order that every matrix

$$a_1 G_{1,i_1} + a_2 G_{1,j_1}, \dots, a_1 G_{1,i_k} + a_2 G_{1,j_k}, \dots, a_1 G_{1,i_n} + a_2 G_{1,j_n}$$

has the same rows then in combination (8) and (10) with the same coefficients a_1 and a_2 there will be a matrix with the same rows as for (8) and (9) (when G_{2,j_j} is an identity matrix, for example) and there will be a matrix with different set of rows when G_{2,j_i} is not equal to any of matrices from (9). This contradicts condition (A).

Theorem 2. If associative semihypergroup \mathbf{H} satisfies condition (A), then it satisfies condition (B).

It has been shown above that every matrix A_i of left regular representation of \mathbf{H} is a combination of G_i - matrices of left representation of some group \mathbf{G} . By (A) any A_i should have the same set of rows that the matrix A_1 has. Let us assume that

$$A_1 = a_1 G_1 + a_2 G_2 + \dots + a_n G_n$$

In the product $G_i A_1$, matrix G_i acts as a permutation of rows in matrix A_1 . So, the only possibility to satisfy the following two conditions: A_i as a combination of G_i and A_i contains the same set of rows as A_1 is

$$A_i = G_i A_1 = a_1 G_i G_1 + a_2 G_i G_2 + \dots + a_n G_i G_n$$

The first means that A_i contains the same rows as A_1 and the second that A_i is a combination of G_i .

The rest should be clear from the following consideration

Theorem 3. If an associative semihypergroup \mathbf{H} satisfies condition (B) then it satisfies condition (A).

Let \mathbf{G} be a group of order n and $g_i \cdot g_j$ denote product operation and m a probability measure on \mathbf{G} . Let us build a semihypergroup \mathbf{H} with elements $\{e_1, \dots, e_n\}$ where $e_i = m \cdot g_i$ and convolution operation as

$$e_i * e_j = (m \cdot g_i) \cdot (m \cdot g_j)$$

Because elements of \mathbf{H} are $m \cdot g_i$, the convolution should be described in terms of $(m \cdot g_i)$.

$$e_i * e_j = m \cdot g_i \cdot m \cdot g_j = m \cdot (g_i \cdot m \cdot g_j) = m \cdot \left(\sum m_{i,j}(k) g_k \right)$$

Let us consider how the i -th matrix M_i of a regular representation of \mathbf{H} is built. Each $m_{i,j}$ the j -th column in M_i is the ordered sequence of coefficients in $g_i \cdot m \cdot g_j$ -

$\{m_{i,j}(k)\}$.

Because

$$m \cdot g_j = m_1 g_1 \cdot g_j + m_2 g_2 \cdot g_j + \dots + m_n g_n \cdot g_j$$

and $g_i \cdot g_j$ is the j -th column in G_i (more precisely j -th column in G_i is the sequence $\{g_{i,j}(1), \dots, g_{i,j}(n)\}$ of coefficients in $g_i \cdot g_j = g_{i,j}(1)g_1 + g_{i,j}(2)g_2 + \dots + g_{i,j}(n)g_n$) we have to conclude that $m \cdot g_j$ represents the column $\sum m_k g_{k,j}$ and $m \cdot g_j (j = 1, \dots, n)$ should be represented by matrix

$$\sum m_k G_k = M = \begin{pmatrix} m_{1,1}(1) & m_{1,2}(1) & \dots & m_{1,n}(1) \\ m_{1,1}(2) & m_{1,2}(2) & \dots & m_{1,n}(2) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ m_{1,1}(n) & m_{1,2}(n) & \dots & m_{1,n}(n) \end{pmatrix}$$

where $m_{1,1}(1) = m_1, m_{1,1}(2) = m_2, \dots, m_{1,1}(n) = m_n$. Now to add scalar multiplier g_i to M , we need to implement the multiplication by component g_i on M . It means that each k -th row in M ($m_{1,1}(k), m_{1,2}(k), \dots, m_{1,n}(k)$) that is $\{m_{1,1}(k)g_i \cdot g_k, m_{1,2}(k)g_i \cdot g_k, \dots, m_{1,n}(k)g_i \cdot g_k\}$ will be moved to p -th row where $g_p = g_i \cdot g_k$.

It means that $M_i = G_i M$. Because in $G_i M$, G_i acts as permutation of rows in M , any M_i has the same set of rows as M .

The same is true with respect to the right regular representation of \mathbf{H} . Next, if measure m is not a uniform measure concentrated on some subgroup of \mathbf{G} , all rows in M_i will be different and linearly independent.

Semihypergroup \mathbf{H} is associative because underlying group \mathbf{G} is associative and $e_i * e_j = (m \cdot g_i) \cdot (m \cdot g_j) = m \cdot (g_i \cdot m \cdot g_j)$.

References

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